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Understanding Prospective Teachers' Levels of Geometric Thoughts: Insights from a Discursive Analysis

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UNDERSTANDING PROSPECTIVE TEACHERS' LEVELS OF GEOMETRIC THOUGHTS: INSIGHTS FROM A DISCURSIVE ANALYSIS

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The study investigates the characteristics of prospective teachers' geometric discourses at the van Hiele model of thinking (1959/1985), using Sfard's (2008) discursive framework. In this report, I align two prospective teachers' pre- and post- van Hiele geometry test (Usiskin, 1982) results with the analyses of their geometric discourses from clinical interviews, to illustrate changes in geometric discourse when a student's test results showed no change in van Hiele levels, and changes in geometric discourse when a student developed her thinking to the next van Hiele level. Revisiting the van Hiele model of thinking, complemented with a discursive lens, helped to understand learning as change in discourses, as prospective teacher develop thinking toward a higher van Hiele level.

Keywords: Geometry and Geometrical and Spatial Thinking, Learning Progressions, Teacher knowledge

The van Hiele model of thinking, known as "the van Hiele levels", was developed by the Dutch educators Pierre and Dina van Hiele (1959/1985). Many researchers have confirmed the usefulness of van Hiele levels when describing the development of students' geometry thinking. However the same researchers often find levels lacking in depth, and they would like a more detailed description of students' levels of thinking. Hoffer's (1981) "Sample Skills and Problems" (p.11) provided a framework that connects the levels of development with five basic skills (e.g., visual skills, verbal skills, drawing skills, etc) that are expected at each van Hiele level. Battista (2007) refined the model with five levels of geometric reasoings. Given my focus on prospective teachers' learning in geometry, I consider the possibility of elaborating the van Hiele model of thinking with a discursive lens in scrutinizing prospective teachers' thinking. I claim that when a student's geometric thinking develops to a higher level, simultaneously there is a development of the student's geometric discourse in discursive terms. If so, the question is, "What additional information does the analysis of geometric discourse provide about prospective teachers' levels of geometric thoughts?"

Theoretical Framework

The van Hiele model continues to be the best-known theoretical account of students' learning of geometric figures and their properties. The model suggests students must progress through a sequence of discrete, qualitatively different levels of geometry thinking. The first four levels in the model are as follows: Level 1, the Visualization level in which students recognize and learn the names of the figures, and figures are judged by their appearance as a whole; Level 2, the Descriptive level, when students begin to recognize figures by their properties or components; Level 3, the Theoretical level, where students begin to form definitions of figures based on their common properties, and understand some proofs; Level 4, the formal logic level, when students understand the meaning of deduction and conduct mathematical proofs using theorems.

My study departs from the van Hiele model, which accepts the basic idea of the levels of geometric thinking and of these main characteristics. In my rendition, thinking becomes a form of communication, and levels of geometric thinking become levels of geometric discourse. This view of geometric thinking as geometric discourse entails that thinking is communicated through interactions.

Sfard (2008) has proposed that mathematical discourses differ one from another in at least four features: 1) *Word use* (mathematical vocabularies and their use), mathematical words that signify mathematical objects or process; 2) *Routines*, these are well-defined repetitive patterns characteristic of the given mathematical discourse; 3) *Visual mediator*s, these are symbolic artifacts related especially for particular communication; 4) *Endorsed narratives,* any text, spoken or written, which is framed as description of objects , of relations between processes with or by objects, and which is subject to endorsement or rejection, that is, to be labeled as true or false. These features interact with one another in a variety of ways. For example, endorsed narratives contain mathematical vocabularies and provide the context in which those words are used; mathematical routines are apparent in the use of visual mediators and produce narratives.

In what follows, the van Hiele behavioral descriptions at each van Hiele level are reviewed and analyzed with discursive terms (see Wang, 2011). For example, at Level 1, "a child recognizes a rectangle by its form, shape" (the van Hiele quotes), provides information about how a child identifies a figure, what it calls "a rectangle", based on its physical appearance. In discursive terms, the *vocabulary,* "rectangle", signifies a geometric shape has a name, and it is used as a label of the figure. The phrase, "recognizes…by its form, shape" suggests that the direct recognition triggers the decision making, and therefore the *routine* for this course of action is perceptual experience and it is self-evident (i.e., [it is] a rectangle [because I can see it] by its form, shape). The *narrative* is "what is said or described" about the object. The *visual mediators* in this situation could include a drawing or picture of a four-sided figure looking like a rectangle. As described, using the discursive lens not only allowed me to capture what students said, but also what they did when communicating their thinking. Viewing each van Hiele level thinking as its own geometric discourse, and with the help of the discursive lens, I expected to be able to arrive at a refined, "high resolution" picture of the process of geometric thinking.

Method

This report focuses on two participants. Sam and Lulu were prospective teachers, enrolled in a certain mid-west university teacher education program in the United States. They were enrolled in a measurement and geometry course as part of their program requirements. They participated in a pre- and a post-van Hiele Geometry Test (see Usiskin, 1982) as class assignments. One week after Sam and Lulu took the pre- and post-tests, they participated in 90-minutes pre-and post-interviews. The interview tasks are designed to elicit students geometric thinking and they are aligned with van Hiele geometric tests (see Wang, 2011). All the interviews are video recorded and transcribed for further discursive analysis.

I analyzed Sam and Lule's written responses from the van Hiele geometry test results using the test grading methods provided by the Chicago project team (Usiskin, 1982) to get initial information on their levels of geometric thinking. I also analyzed Sam and Lulu's interview transcripts in both what they said and what they did as course of actions during the interviews. In the following section, I share results about Sam and Lulu's test results and their interview analyses to learn more about the process of geometric thinking.

Results

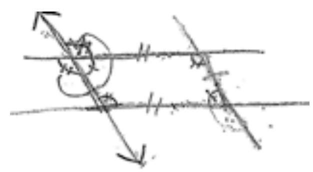
Viewing the van Hiele model with discursive terms sheds light on what vocabularies participants used in describing geometric figures, and how these vocabularies were used in communicating mathematical thinking. The discursive lens helped to bring participants' actions to light, making it possible fro me to differentiate how they substantiated their narratives, either at a meta-level or at an object-level.

Continuity Within A Van Hiele Level: Sam's Case

Sam's van Hiele tests suggested that she was at Level 2 at the pre-test and stayed at Level 2, with no changes in van Hiele levels, but I found changes in her geometric discourses. The analyses of Sam's geometric discourses showed that her use of the word parallelogram changed. When she spoke the word "parallelogram" at the pre-interview, she meant any polygon having pairs of parallel sides, using a definition of parallelogram with only a necessary condition. For example, Sam was asked to draw a parallelogram and then a new parallelogram different from the previous one. Sam's course of action is as follows:

Interviewer: Why is this a parallelogram? [Pointing at the parallelogram]

Sam: I drew it so that this side would be parallel to this side [pointing at the two longer sides of the parallelogram], and this side would be parallel with this side [pointing at the two shorter sides of the parallelogram] Note: Sam drew a parallelogram first, and extended the sides of the parallelogram later.

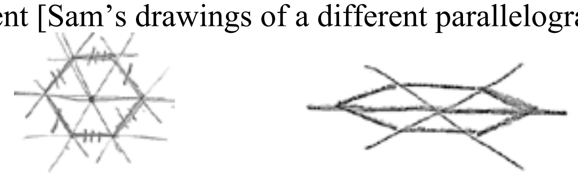


Interviewer: Why is this a parallelogram? [Pointing at the hexagon]

Sam: … because all the sides are parallel to another side.

Interviewer: Why is it a different parallelogram?

Sam: It's different because there are more sides and because the angles are different [Sam's drawings of a different parallelogram].

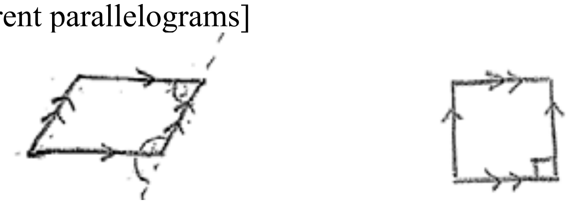


When communicating about parallelograms at the pre-interview, Sam considered two types: 1) "a parallelogram is a figure with all sides being part of parallel line segments (i.e., rhombi, parallelograms, hexagons, octagons, etc.); and 2) "it is a figure that has at least one pair of parallel sides, I think a trapezoid is a parallelogram". Sam did not consider squares and rectangles as parallelograms at the pre-interview.

At the post-interview, the same task was performed and Sam responded to the task with two drawings, a parallelogram and a square.

Interviewer: Why is this a parallelogram? [Pointing at the parallelogram].

Sam: … because it has four sides and each opposing side is parallel to one another. [Sam's drawings of two different parallelograms]



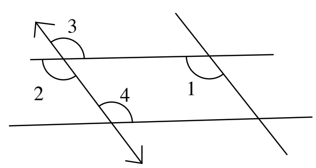
Interviewer: Why is this a parallelogram? [Pointing at the square]

Sam: It's a square, square is also a parallelogram, it has four equal sides and all angles are 90 degrees [Pointing at the square]. It's a different parallelogram from the one I drew because all the angles in this figure are equal.

It is notable that Sam's use of the word parallelogram evolved from pre-interview to post-interview, showing changes in her understanding of parallelograms, with regard to the added necessary conditions "four-sided" figure and "parallel sides". Moreover, Sam's grouping of parallelograms showed changes at the post-interview. She grouped parallelograms and rhombi together because "they all have two sets of parallel sides". She asserted, "rectangles are parallelograms with four right angles", as well as "squares are parallelograms with four right angels and four sides are equal." I argue that, at the post-interview, Sam had a good grasp of the concept of parallelograms in general, but her understanding of the hierarchy of parallelograms was missing, or not clearly demonstrated in the interviews.

There were also changes in Sam's substantiation routines. Substaintiation routines are repetitive patterns charaterising how Sam justifies or proves that naratives she provides are true or false. For example, at the pre-interview, Sam frequently used reflections, rotations and translations in her substantiations of narratives as concrete descriptions of her investigations.

Interviewer: How do you know ∠1 is equal to ∠4?



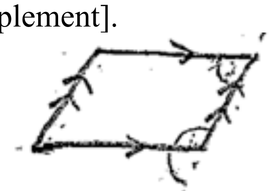
Sam: This angle [∠1] can just be slid over to this position and create ∠2, this line (the one with arrowheads) can be rotated so that ∠2 becomes ∠3. This angle [pointing at ∠3] at this intersection can just slide down and be in this angle's position [pointing at ∠4]. So two angles (∠1 and ∠4) are equal.

In this explanation, Sam used words such as "slide over", "create", "rotated" and "slide down" to indicate a sequence of imaginary movements performed to substantiate the narrative "two opposite angles [∠1 and ∠4] are equivalent". Sam's substantiation was intuitive and visual, and was focused on the descriptions of how lines and angles moved, rather than on the discussions about the results. Sam's routines operated at an object-level at the pre-interview.

In contrast, at the post-interview, Sam was able to use endorsed narratives (i.e., mathematical axioms and propositions) to verify her claims. The following brief substantiation was typical at the post-interview:

Interviewer: How do you know all angles in a parallelogram add up to 360 degrees?

Sam: Because angles on a straight line are going to add up to 180 degrees [Pointing at ∠2 and its complement].



Sam: This angle here is the same as this angle [pointing at ∠2 and its transversal exterior angle] because the parallel lines meet a third line at the angle. By the

> same reason, this angle added to this angle [pointing at ∠1 and ∠2's transversal exterior angle] equals 180 degrees. These two angles [∠1 and ∠2] added up to 180 degrees. For similar reasons, these two [opposite angles of ∠1 and ∠2] added up to 180 degrees. Together they [all four angles of the parallelogram] equal to 360 degrees.

Sam explained why two transversal angles are equivalent, not because you "can see it" as in the pre-interview, but as a result of "two parallel lines meet a third line at the same angle", and reached her conclusion that " they [all four angels of the parallelogram] equal to 360 degrees" by repeating a similar substantiation, "two angles add up to 180 degrees", for two adjacent angles in a parallelogram.

Note that the term "geometric object(s)" refers to all the mathematical objects involved in a particular geometric discourse. In my study, geometric objects discussed are quadrilaterals. The term "substantiation" refers to substantiations at an object-level and at a meta-level. The object-level substantiation emphasizes students' routines in describing the process of how quadrilaterals are investigated. For example, describing static lines, angles and polygons as movable entities under transformations (i.e., rotation, translation and reflection), is a way of substantiation at an object-level. With regard to definitions of different quadrilaterals, routines of substantiation depending on measurement routines to check the sides and angles of quadrilaterals, without thinking about how quadrilaterals are connected, are other examples of an object-level of substantiation. Object-level substantiation is a course of action where the student focuses on what she sees intuitively in explaining geometric objects (i.e., quadrilaterals) during the investigations. In contrast, a meta-level substantiation is a course of action where the student uses endorsed narratives to endorse new narratives. That is, students use mathematical definitions and axioms as results of investigations to construct mathematical proofs.

Two main changes in Sam's geometric discourse are change in word use and change in routines. Sam had developed competence in using definitions to identify and to group polygons with no hierarchy of classification, and had developed some informal deductive reasoning as her geometric thinking moved towards Level 3. Here, I am not trying to contradict the findings from Sam's van Hiele Geometry tests with her interview results, but rather to compile the results and to treat her thinking more explicitly and dynamically. The development of Sam's geoemtric discourse provides evidence of a student's geometric thinking developing continuously within Level 2 and in transition between Level 2 and Level 3, as she became more competent in using definitions to identify polygons, and her routines of substantiation began to operate at a meta-level in using definitions and axioms to construct mathematical proofs.

Continuity Within Two Consecutive Levels: Lulu's Case

Lulu was one of the two prospective teachers of the study who reached Level 4 based on the van Hiele geometry post-test. Lulu was at Level 3 in the pre-test and demonstrated a typical behaviour at this level. So one might ask, "what did Lulu's geometric discourse look like in moving from Level 3 to Level 4?" The detailed analyses of changes in Lulu's discourses are documented in my dissertation (see Wang, 2011). I summarize the changes briefly here. Lulu was at Level 3, coming in with the ability to use definitions to identify and group quadrilaterals, but my analyses of Lulu's geometric discourse show that she did not demonstrate that quadrilaterals were connected in a hierarchy of classifications. For example, in the pre-interview, Lulu's use of the word *parallelogram* referred to two groups: 1) rhombi and parallelograms, characterized by "opposite sides are equal and parallel" and 2) rectangles and squares, characterized by "opposite sides are equal and parallel, and they all have right angles". Lulu did consider squares as rectangles, but she made no connections between rhombi and squares. At the pre-interview, Lulu performed an object-level of substantiations. For instance, when discussing the angles of a parallelogram, Lulu responded that the opposite angles of a parallelogram were equal. The following conversation took place when she was asked for verification:

Interviewer: How do you know the opposite angles are equal [in a parallelogram]?

Lulu: You mean… prove it to you, that in every case it would be that way?

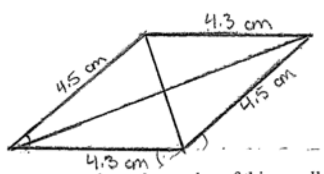
Interviewer: Yeah.

Lulu:I would just measure the angles for you, with a protractor. I've never done a proof before, in this case. I've done a lot of proofs, but not on something like that.

In this conversation, we learned that writing a geometry proof was new to Lulu, but she was aware of the differences between the generality of a mathematical proof and the particularity of checking the measurements of angles in a parallelogram when she asked, "prove it to you, that in every case it would be…?" In a similar scenario, Lulu measured the sides of parallelograms to verify her claim of, "the opposite sides in a parallelogram are equal".

Interviewer: How do you know opposite sides are equal?

Lulu: In this parallelogram? I can measure it. So, this is 4.5 centimetres, this is a little less than 4.5 centimetres [using a ruler to measure one pair of opposite sides]. Right, this is about 4.3. Yeah, it's about the same [measuring another pair of opposite sides].

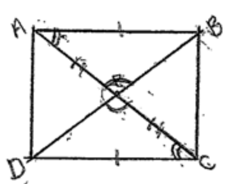


Interviewer: Is it true for every parallelogram?

Lulu: You mean to prove it? Well, I am not sure… but I know it's just a property of parallelogram.

The patterns of measuring sides and angles to verify equal measures were apparent in the pre-interview. In contrast, Lulu was able to demonstrate the hierarchy of classification among quadrilaterals using definitions at the post-interview, and she used propositions and axioms to construct mathematical proofs at that time. For example, Lulu provided a narrative, "diagonals bisect each other in a rectangle", and I asked for substantiation.

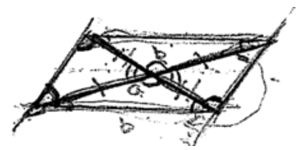
Interviewer: How do you know diagonals bisect each other in this case [pointing at Lulu's drawing, rectangle]?



Lulu: For the same reason as last time. Do you want me to explain again?

Interviewer: When you said, "for the same reason as last time", what do you refer it to?

Lulu: Just, all of it. When you draw diagonals in a rectangle, the diagonals create these congruent triangles. Based on the property, in all parallelograms, diagonals bisect each other. I know that the diagonals bisect each other in rectangles [pointing at the congruent triangles in the parallelogram].



The previous conversation signalled Lulu's ability in making relational transfer, from an endorsed narrative of "in all parallelograms diagonals bisect each other" to justify her claim about the diagonals in a rectangle. Given this opportunity, I asked for a written proof. Lulu's written proof is presented in Figure 1.

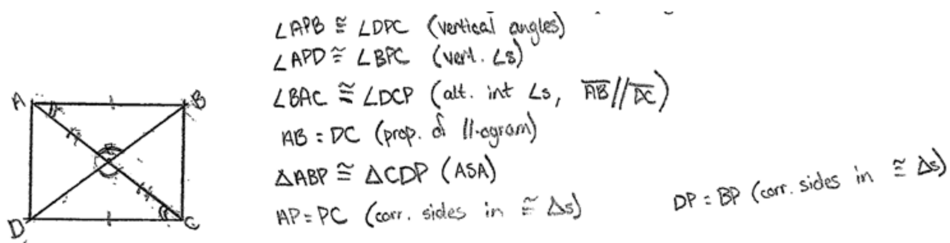


Figure 1: Lulu's Written Proof that Diagonals Bisect Each Other in a Rectangle at the Post-Interview

Lulu's geometric discourse presents a main characteristic of a Level 4 discourse that is absent in Level 3: abstraction of substantiations. Lulu showed Level 4 thinking in using definitions and axioms to construct proofs, and in using algebraic symbols to write a formal mathematical proof. However, at this level we also expect to observe behaviour where students are able to apply inductive reasoning in an unfamiliar situation. In Lulu's case, she was able to apply her knowledge of quadrilaterals to construct mathematical proofs in a familiar situation (e.g., to prove opposite angles or sides are congruent using congruent criterions), having carried out similar proofs in her geometry class for elementary teachers. When Lulu was asked to prove that two definitions of parallelogram were equivalent, she did not finish the proof because the task was new to her, and she did not know how to use the same axioms in a new situation. I argue that Lulu was at the beginning of Level 4 thinking, starting to gain the skills and languages needed for mathematical proof, but needing more practice to move forward to an advanced abstract level. Examining the changes from Lulu's Level 3 geometric discourse at the pre-interview to Level 4 at the post-interview, we observe movement from Level 3 thinking to Level 4 in developing a geometry discourse, in a continuous progression instead of jumps.

In addition, comparing Lulu's geometric discourse at Level 3 with Sam's at Level 2, we find similarities between Lulu's geometric discourse at the pre-interview and Sam's at the post-interview. Both shared familiarity with using definitions in identifying and grouping quadrilaterals, and were able to reason at an object-level. This observation also indicates the continuity of learning in transitioning between two consecutive levels, from Level 2 to Level 3.

Conclusions

The discursive framework provided opportunities to examine students' thinking in greater detail at each van Hiele level. It helped to analyse what students said (i.e. narratives and word use) about different parallelograms and their properties, and what they did (i.e., routines) when asked for substantiations. A careful analysis of students' mathematical word use sheds light on how words are used and whether the words are used correctly. Discursive routines do not determine students' actions, but only constrain what they can reasonably say or do in a given situation, as negotiated conventions. However, discursive routines offer valuable information about what students do to make conjectures and justifications in a geometric discourse. I find it very useful to examine the details of students' routines of identifying, measuring, defining and justifying when working on a task about geometric figures and their properties, where the roles of definitions are demonstrated at the first three van Hiele levels. I also find it revealing to see the details of students' geometric reasoning across van Hiele levels through the development of geometric discourses.

Battista (2007) argues about the validity of the reasoning, which involves the accuracy and precision of students' identifications, conceptions, explanations, justifications, and points out that "there is a lack of distinction between *type of reasoning* and *qualitatively different levels in the development of reasoning*" (p.853) throughout van Hiele studies. For instance, a student used direct recognition as a type of reasoning that is strictly based on intuition, and used the same type of reasoning to refer to a period of development of geometric thinking when the student's thinking was dominated by direct recognition. One challenge regarding the van Hiele model is to sort out the levels related to type of reasoning and/or the levels of reasoning; and of course, "the devil is in the details" (p.854). Viewing geometric thinking as geometric discourse, Sfard's discursive framework takes a greater consideration of the details.

Acknowledgment

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